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Yannick Meurice

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Making digital aquatint with the Ising model

Yannick Meurice^{a)}

Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242

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Aquatint is a printmaking technique that gives visual texture to prints. This patterning is shown to be similar to patterns that emerge from the Ising model of magnetism. We provide a concrete Python-based example that generates semi-realistic textures and can be used to manipulate existing images. Our experience in classrooms of first year seminar undergraduate students shows that this activity helps artistically inclined students to develop an interest in physics and computer science. It may also be suitable for a joint art and science project for high-school seniors. © 2022 Published under an exclusive license by American Association of Physics Teachers.

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I. INTRODUCTION

The statistical features of images or image sets are very interesting. High resolution black and white digital pictures of prints can capture many aspects of the original image and encode the information into a collection of pixels having gray values between 0 and 255. Conversely, visual aspects of statistical data can play an important role in identifying key features in samples. Can we use concepts and models of statistical mechanics to reproduce artistic features of images and attempt to quantify variations observed in these images? For instance, in visual arts like printmaking, the concept of *texture* plays a crucial role in the creative process and in the description of existing prints. It is not easy to define texture in a scientific manner, but in casual conversations, printmakers sometimes talk about the texture as the way the printed surface would feel to the touch and they use qualifiers such as rough, grainy, or smooth. Is it possible to invent statistical mechanics models that approximately reproduce such features? In order to simplify the discussion, we will assume that in the first approximation, the pixels are either black or white and carry only one bit of information. This simplified approach suggests that the Ising model¹ could be used to reproduce textures achieved in printmaking or manipulate features of existing images.

The Ising model is a highly simplified model of ferromagnetism. It consists of “spin variables” σ_x that only take the values ± 1 and are located on the sites x of a lattice. The physical meaning of the spin variables is discussed in more detail below. In the following, we only consider lattices that can be built by assembling elementary squares of identical size in rectangular arrays because of their clear connection to pixelated pictures. Similar constructions are possible for triangular, hexagonal or higher-dimensional lattices. We can visualize a particular choice of spin values as a collection of black or white pixels assembled in a two-dimensional rectangular array. We call this collection a configuration. In this representation, the sites of the lattice are the coordinates of the center of the pixels. An example using a 50×50 lattice is displayed in Fig. 1(a). The black pixels represent $\sigma = +1$ (spin up) and the white pixels $\sigma = -1$ (spin down).

In modeling physical systems, the spin variables represent magnetic dipole moments or the intrinsic spins of electrons and are often depicted as arrows pointing up or down. The Ising model is a statistical model where the spin configurations have a classical Boltzmann probability distribution, which depends on the temperature and an external magnetic

field and involves nearest neighbor interactions that correlate spins in the distribution. The temperature controls the amount of disorder appearing in the system while the magnetic field is used to introduce order. At low temperature, domains, which can be visualized as “islands” of spins having the same sign, are more likely to appear. As we increase the temperature, these domains tend to break into smaller ones. At high temperature, neighboring spins become uncorrelated, as in Fig. 1(a). This intuitive description of the effect of the temperature will be made more quantitative later.

In Fig. 1(a), the individual pixels (spins) are still quite visible. However, if we increase the number of pixels while keeping the size of the image constant, and introduce a magnetic field and a temperature, textures that artists may consider interesting can be generated. An example is shown in Fig. 1(b). In this example, a uniform positive magnetic field has been applied. This favors black pixels (spin up) over white pixels (spin down). This procedure can be refined by adding a spatially modulated magnetic field to generate a recognizable image. This will help us to address the question of the texture of pixelated images in the framework of the Ising model.

In the following, we show that it is possible to tune the adjustable parameters (temperature and magnetic field) of the Ising model, and use a statistical method called importance sampling, to generate configurations with textures approximating those obtained with a printmaking technique called aquatint, which was developed by artists including Francisco Goya, Pablo Picasso, and Mauricio Lasansky. A specific print of Goya is discussed in Sec. II.

The basic aspects of the Ising model are reviewed in Sec. III. There is a critical temperature below which the system

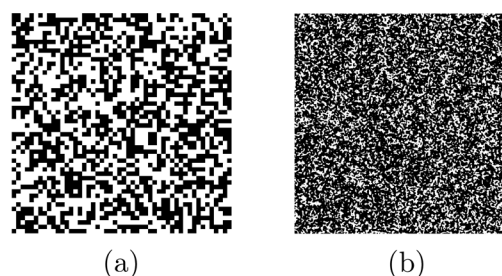


Fig. 1. (a) Example of a random Ising configuration on a 50×50 square lattice. The orientations of the spins are uncorrelated as in a high-temperature situation with no magnetic field applied. (b) Configuration on a 1000×1000 square lattice. Ten iterations of the code described in the text were used with a dimensionless temperature $T = 0.5$ and a uniform magnetic field $h = 0.1$.

magnetizes in the presence of an arbitrarily small magnetic field. This is an example of a *second-order phase transition* which is of great interest in the context of statistical mechanics. While a detailed understanding of second-order phase transitions is not essential for understanding the rest of this article, scientifically inclined students may be interested in interpreting images from this perspective.

In the absence of an external field breaking the symmetry between up and down, configurations tend to be without recognizable shape or features. However, it is possible to use a pixelated picture to define a magnetic field h_x for each pixel, and then use h_x along with an assigned temperature T to add texture to the original picture with a procedure called importance sampling. This procedure is described in Sec. IV and by using it, the pixels of a configuration are more likely to be as in the original picture but with fluctuations that provide an additional texture. In other words, the procedure that we describe randomizes the original picture by introducing thermal fluctuations. As an example, a modified picture of Albert Einstein, with black and white pixels, is shown in Fig. 2. The details of the method used to generate Fig. 2 are discussed in Sec. V. In the following, we will consider a spatially uniform temperature. Introducing local variations of the temperature would be an interesting extension of the process described here.

The code generating Fig. 2 is relatively simple. Some of its lines can be manipulated by students having no programming experience. Students are able to get interesting results by simply changing the temperature or the strength of the magnetic field, as summarized in practical terms in Sec. VI. This has been a successful experience for students of several first-year seminars. Some artistically inclined students, who thought that they had no interest in computing, discovered the pleasure of “looking under the hood” to adapt the existing code to achieve their personal artistic goals.

II. AQUATINT

Aquatint is a printmaking technique, where particles of rosin or a material with similar semi-porous properties are melted on the printing plate in order to protect the covered parts from the corrosive etching agent. This leaves small uncovered patches forming intricate microscopic patterns, which will be partially removed during the etching process. After etching, the resist is removed and ink is applied. The ink collects in the patches removed from the metal, while the covered parts stay smooth and free of ink after the plate is



Fig. 2. Example of Ising configuration obtained with a temperature 1.9 and a magnetic field provided by a public domain picture of Albert Einstein used according to Eq. (3) with $h = 1.0$.

carefully wiped. This produces a very rich texture that can be seen in most of the background of plate 59 of Goya’s “The Disasters of War” describing the desperation of survivors of the French–Spanish war around 1812. The full print in true color is shown in Fig. 3. The actual size of the central plate is about $13 \times 18 \text{ cm}^2$.

A higher resolution grayscale image of the upper-left corner of the plate was kindly provided by the University of Iowa Museum using their own copy of plate 59.⁵ The texture can be seen better on a magnification of the upper left corner of the print shown in Fig. 4. The size of this upper left corner is approximately $8 \times 12 \text{ cm}^2$ and contains 2677×4017 pixels. The size of a pixel on the picture with the original scale is approximately 0.03 mm.

We extracted a 1000×1000 pixel subset corresponding to pixels 500–1500 in both directions, starting with the upper left corner as (0,0). We then converted the eight bit grayscale (0–255) to units going from 0 to 1 and applied a gray cut of 0.4 to convert the grayscale picture into a spin configuration: The pixels below 0.4 become white and the rest of the pixels are black. This is shown in Fig. 5(a). The aquatint process relies on the fact that rosin arranges itself in interesting random patterns. This is a complicated physical process that can, in principle, be modeled using statistical mechanics, fluid dynamics, chemistry, etc. However, it is possible to create macroscopically similar effects simply by using the Ising model. An example of a configuration obtained after 20 iterations of the procedure described in Sec. V for $T = 0.1$ and a constant magnetic field $h_x = h = 0.1$ is shown in Fig. 5(b). It appears to have a texture resembling the texture of the part of the print shown in Fig. 5(a).

III. THE ISING MODEL

The Ising model¹ has many appealing features from a theoretical physics point of view. One of them is the prediction of a second-order phase transition between an ordered and disordered phase at a critical temperature T_c . At high temperature ($T > T_c$), the spins tend to be independently random and in the absence of an applied magnetic field no macroscopic magnetization is present. At low temperature ($T < T_c$), the spins tend to align with a very weak magnetic field. This is often called “spontaneous magnetization.” At



Fig. 3. F. Goya, The Disasters of War, Plate 59. The jpeg file is public domain: <https://images.metmuseum.org/CRDImages/dp/original/DP817452.jpg>.



Fig. 4. Upper left corner of plate 59 shown in Fig. 3. Photo provided by University of Iowa Stanley Museum of Art (Refs. 5 and 6).

the critical temperature T_c , a second-order phase transition takes place and the magnetic susceptibility diverges. In this section, we provide more details about this feature; however, a detailed understanding of second-order phase transitions is not necessary to follow the practical steps discussed in the coming sections.

Our starting point to discuss the phase diagram of the model is the Boltzmann probability distribution. The probability for a given spin configuration $\{\sigma\}$ (the symbol $\{\}$ means the collection of all the spins) will be written as

$$\text{Prob}(\{\sigma\}) = \exp \left(\beta \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \sigma_{\mathbf{x}} \sigma_{\mathbf{y}} + \sum_{\mathbf{x}} h_{\mathbf{x}} \sigma_{\mathbf{x}} \right) / Z, \quad (1)$$

where $\beta = 1/T$ is the inverse temperature, $h_{\mathbf{x}}$ is a local magnetic field, both in appropriate units discussed below, and $\langle \mathbf{x}, \mathbf{y} \rangle$ means pairs of nearest neighbors spins (adjacent pixels) on the lattice. We define the partition function Z to be the sum of this exponential over all the configurations

$$Z = \sum_{\{\sigma\}} \exp \left(\beta \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \sigma_{\mathbf{x}} \sigma_{\mathbf{y}} + \sum_{\mathbf{x}} h_{\mathbf{x}} \sigma_{\mathbf{x}} \right). \quad (2)$$

A few words regarding signs and units: The energy is lower when neighboring spins are aligned with each other or when a spin is aligned with its local magnetic field. Consequently, both energy terms have a minus sign which

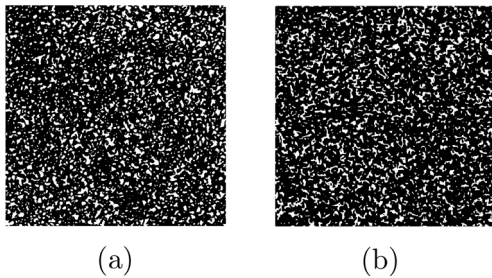


Fig. 5. (a) The 1000×1000 pixel detail of Fig. 4 described in the text with a gray cut of 0.4. The squares have approximately the same size as on the actual print ($3 \times 3 \text{ cm}^2$). (b) The 1000×1000 pixel configuration obtained from the Ising model after 20 iterations of the procedure described in the text for $T = 0.1$ and a constant magnetic field $h = 0.1$.

cancels the minus sign in the exponent of the Boltzmann distribution. The spin-spin interaction provides an energy scale that we have set to one and β is the inverse temperature in these units. In addition, the magnetic field should be understood to be in units of the temperature (so β does not factorize in the exponent). With these conventions, all the quantities are dimensionless and the two couplings can be manipulated independently. Notice that β is the same for every pair of nearest neighbors. Relaxing this constraint could lead to interesting variations.

The magnetization is defined as the spatial average of the spin, itself averaged with the probability given in Eq. (1). It can be obtained by taking the *first* derivative of the logarithm of the partition function with respect to a constant magnetic field. For this reason, a discontinuity in the magnetization is called a *first-order* phase transition.

Configurations where many neighboring spins have the same orientation have lower energies and are more probable than configurations where many neighboring spins have opposite orientations. The typical size of the domains with a given spin depends on the temperature. As we mentioned above, the model has two phases. At high temperature, the spins are essentially uncorrelated (disordered phase) and at low temperature they tend to form clusters of $+$ or $-$ spins (ordered phase). For a temperature below T_c , the magnetization changes sign abruptly when we change the sign of a uniform magnetic field at fixed temperature. As explained above, this is called a first-order phase transition. A second-order phase transition appears at the end of a line of first-order transitions. This is illustrated in Fig. 6. The discontinuity of the magnetization is obtained in the infinite volume limit. At finite volume, the magnetization actually changes rapidly but continuously. Figure 6 was made with a 64×64 lattice.

At a second-order phase transition, the distance at which fluctuations of the magnetization are correlated becomes infinite.² This is a situation of great theoretical interest because universal behavior, independent of microscopic details, is observed. Mathematically, the model at zero magnetic field has an exact solution in one and two dimensions for arbitrary size.³ Figure 6 has been obtained with the code used in the rest of this article to generate typical configurations. The method is discussed in Sec. IV. It is a good exercise to try to

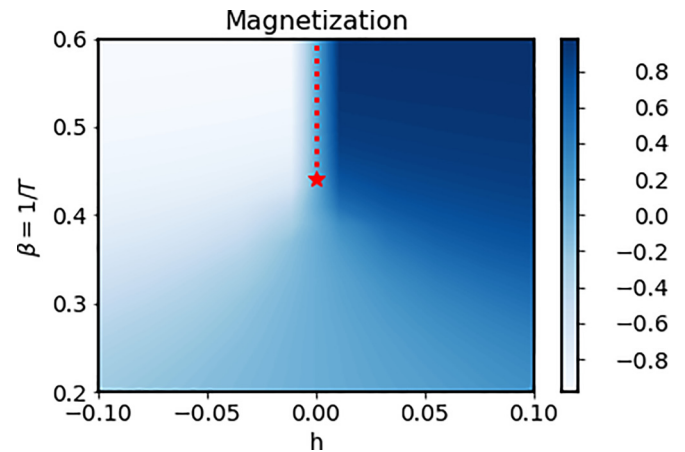


Fig. 6. Magnetization as a function of $\beta = 1/T$ and a uniform magnetic field h . The star denotes the critical temperature and the dots the line of first-order transitions where the magnetization changes abruptly. The graph was made with a 64×64 lattice.

reproduce the phase diagram. Another interesting exercise is to prepare a configuration with a positive magnetic field and then switch to a negative magnetic field and follow the formation of droplets of the opposite magnetization for successive configurations.

In the following, we use rectangular lattices tiled with squares (pixels). Nearest neighbor spins correspond to pixels sharing an edge. We will use magnetic fields of the form

$$h_x = h\sigma_x^{Image}, \quad (3)$$

where the configuration σ_x^{Image} corresponds to an image with black and white pixels translated in spin values with the same convention that we used to visualize configurations (spin-up = black pixel). The adjustable parameter h allows us to tune the strength of the magnetic field with respect to the interaction between spins. One can further visualize the magnetic field as produced by a layer of frozen spins σ_x^{Image} located “on top” of the fluctuating spins and coupled to them by a “vertical inverse temperature” h .

IV. IMPORTANCE SAMPLING

This section describes a statistical procedure to calculate efficiently average quantities with a reasonable accuracy and has been used to generate Fig. 6. This section is intended for readers and students who would like a more detailed account of statistical mechanical simulation, but it can be skipped without loss of continuity.

The calculation of the partition function Z involves a sum over all the possible configurations. In Fig. 1(a), we showed one out of the $2^{2500} \sim 4 \cdot 10^{752}$ possible configurations. Performing this sum in a human timescale is clearly far beyond the reach of any existing computer. However, it is possible to pick a small number of “typical” configurations such that the thermodynamic behavior of the model, for instance, the average energy or magnetization, can be obtained with reasonable accuracy by just averaging the values of the thermodynamical quantities over a small sample of “important” configurations.

A method for generating such typical configurations is to start with a completely random configuration, where each spin is chosen at random without taking into account the other spins (a “hot” configuration) or alternatively where all the spins are the same (a “cold” configuration). We then proceed to flip the spin at a site chosen at random. If the change increases the probability calculated with Eq. (1), the change is always accepted; otherwise the change is accepted with a probability equal to the new probability divided by the old probability. By repeating this process enough times, we end up producing configurations that are important if we wish to calculate, for instance, the average energy or magnetization. As the change in an individual spin has a negligible effect on averages, we usually “sweep” the entire lattice following the procedure described above at each site. We can go through all the sites following a deterministic path. We can also pick the sites at random. With this choice, some sites may be visited more than once and others skipped if we randomly pick L^2 sites for a square lattice of linear size L .

After a complete sweep of the lattice, the new spin configuration can be substantially different from the previous one and is kept in memory. The lattice sweep can be repeated N times. We denote the spin configurations $\{\sigma\}$ generated after each sweep C_i with $i = 1, \dots, N$. The average of a function

of the spins $A(\{\sigma\})$ defined by inserting A in the partition function Z and dividing by Z

$$\langle A \rangle = \sum_{\{\sigma\}} A(\{\sigma\}) \exp \left(\beta \sum_{\langle x,y \rangle} \sigma_x \sigma_y + \sum_x h_x \sigma_x \right) / Z \quad (4)$$

can be approximated by

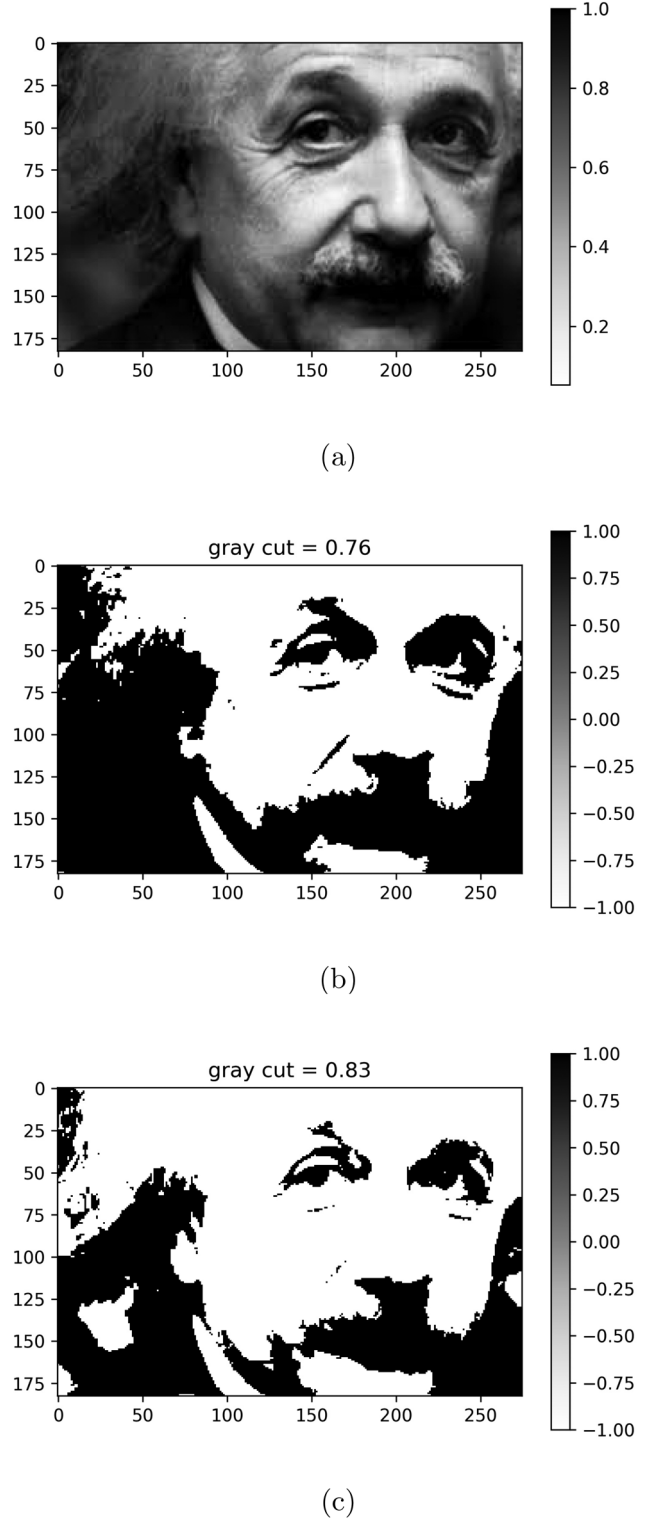


Fig. 7. Original picture with a gray scale between 0 and 1 (a), and the magnetic field obtained with a gray cut at 0.76 (b) and 0.83 (c).

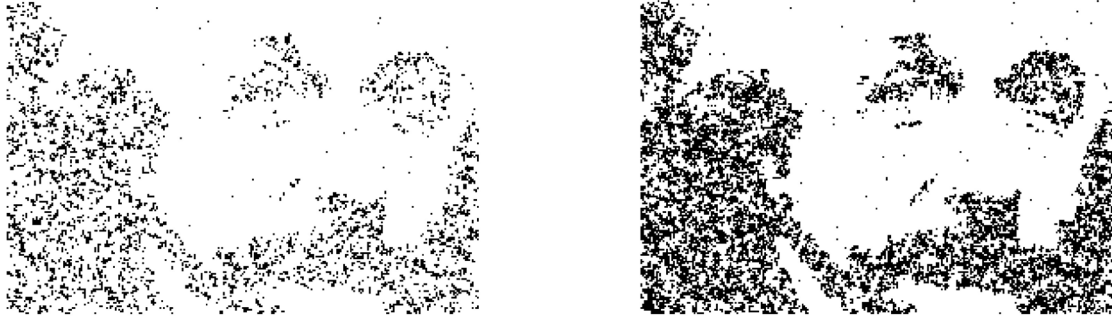


Fig. 8. First two iterations described in the text.

$$\langle A \rangle \simeq \frac{1}{N} \sum_{i=1}^N A(C_i). \quad (5)$$

This is called importance sampling. The convergence of this procedure when N increases has non-trivial aspects. More explanations and references can be found in Ref. 4. The efficiency of this type of method is often a source of amazement for physicists, chemists, and mathematicians. Note also that except for making Fig. 6, averages are not used here and we will now focus on the graphical features of individual configurations.

A remark about randomness. A common procedure for accepting or rejecting changes with a probability p is to generate numbers that are approximately independent and uniformly distributed between 0 and 1. If the number is less than p , the change is accepted, otherwise, it is rejected. Often the computer algorithm generating these numbers is deterministic and the sequence can be reproduced for checking purposes. Typically, they have long cycles and weak correlations among successive sequences of the same length. We call the numbers generated this way, “pseudorandom numbers.”

V. MODIFYING PICTURES WITH THE ISING MODEL

In this section, we use the Ising model with a magnetic field h_x borrowed from a jpeg picture to produce spin configurations sharing features with the original picture. For speedy computation, it is useful to select a picture with fewer than 300 pixels in each direction. An example is given in Fig. 7(a), where the actual size is 183×275 pixels. For pictures of this size, the sample code produces pictures in just a few seconds on most laptops. This allows multiple quick trial-and-error experiments. First, we convert the grayscale picture into a black and white picture, which can be interpreted as an Ising configuration. In Fig. 7(b), we converted the pixels of the original image by assigning those with a grayscale less than 0.76 to white, giving them a value $h_x = -h$, while the other pixels were converted to black and given a value $h_x = +h$. This procedure was repeated with a gray cut of 0.83, producing the image shown in Fig. 7(c).

We then used the importance sampling procedure from the Python code provided in the supplementary material.⁷ The adjustable parameters are T (temperature), the number of sweeps (totalsweeps), and the strength of the magnetic field (sth). The choice sth=1 corresponds to $h=1$ in the previous paragraph. (An example with temperature=1.9, totalsweeps=3, and sth=1 was shown in Fig. 2.) The first two iterations are shown in Fig. 8, where

the initial configuration was totally white. An alternative is to use a random choice for each pixel.

VI. PRACTICAL ASPECTS

Students can use the PYTHON code provided in the supplementary material, for instance, by copying and pasting the cells in a Jupyter notebook. The jpeg image (here publiceinstein.jpg) should be put in the same directory as the PYTHON code. The cells should be activated (usually with shift-return) in the order they are provided. The basic choices of parameters are

- `graycut`: black and white separation with a value between 0 and 1.
- `temperature`: a low value favors clumping; a large value favors small scale fluctuations. Note that the second-order phase transition appears³ at $T = T_c = 2 / \ln(1 + \sqrt{2}) \simeq 2.2692$ with $h=0$.
- `totalsweeps`: number of configurations generated.
- `sth`: overall strength of the magnetic field associated with the original image.

Variations can be obtained by uncommenting some lines:

- use of the original gray magnetic field,
- start white or random, and
- save the picture with a different dpi value.

Questions of artistic appropriation and the boundary of fair use of copyrighted material are complicated⁸ and clearly beyond the scope of this article, so it is recommended to use images that are public or owned by students.

VII. CONCLUSIONS

We have shown how to create textures resembling those obtained with aquatint, by using the Ising model with a magnetic field borrowed from an existing jpeg picture, and then applying importance sampling updates. Students of various backgrounds can make numerical experiments with interesting visual appeal that illustrate phase transitions while developing a taste for numerical simulations.

Many variations can be added to this activity, such as local variations of the temperature or introduction of more possibilities at each site, as in clock models,⁹ which could be translated into choices of color. The self-similar aspects of the configurations near the critical temperature² could be explored and compared with features obtained with other methods used to alter boundaries in images such as dithering or Laplacian sharpening. Finally, the activity can lead to interesting discussions on the role of randomness in art and the creative processes.

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^{a)}Electronic mail: yannick-meurice@uiowa.edu

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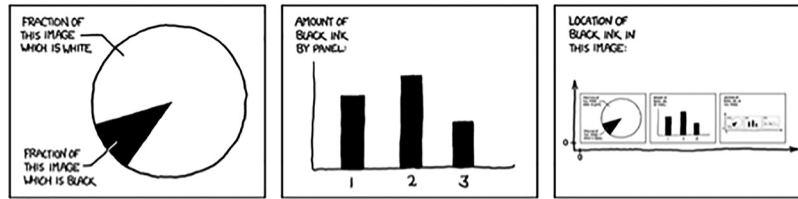
⁵Francisco de Goya y Lucientes, “Plate 59. ¿e qué sirve una taza? (What good is a single cup?) from Los Desastres de la Guerra (The Disasters of War) (detail), 1863 Etching and aquatint,” $9\frac{1}{2} \times 13\frac{1}{8}$ in.2.(24.13 \times 33.34 cm2). Gift of Owen and Leone Elliott, 1976.42BG, University of Iowa Stanley Museum of Art. The photograph was taken by Steve Erickson.

⁶Note that the picture in the public version of the article is a compressed jpeg file with 1654×1102 pixels. The original tif file requires more than 30 MBs.

⁷See supplementary material at <https://www.scitation.org/doi/suppl/10.1119/10.0006525> for the code used to produce the pictures of Einstein.

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Self-Description

The contents of any one panel are dependent on the contents of every panel including itself. The graph of panel dependencies is complete and bidirectional, and each node has a loop. The mouseover text has two hundred and forty-two characters. (Source: <https://xkcd.com/688/>)